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Ghosts in massive gravity

Paolo Creminelli,[a] **Alberto Nicolis,**[a] **Michele Papucci**[b] **and Enrico Trincherini**[c]

[a] *Jefferson Physical Laboratory, Harvard University*
Cambridge, MA 02138, U.S.A.
[b] *Department of Physics, University of California*
Berkeley and Theoretical Physics Group, Ernest Orlando Lawrence Berkeley National Laboratory, Berkeley, CA 94720, U.S.A.
[c] *Institute for Theoretical Physics, Heidelberg University*
D-69120 Heidelberg, Germany
E-mail: creminel@bose.harvard.edu, nicolis@fas.harvard.edu, papucci@berkeley.edu, e.trincherini@thphys.uni-heidelberg.de

Abstract: In the context of Lorentz-invariant massive gravity we show that classical solutions around heavy sources are plagued by ghost instabilities. The ghost shows up in the effective field theory at huge distances from the source, much bigger than the Vainshtein radius. Its presence is independent of the choice of the non-linear terms added to the Fierz-Pauli lagrangian. At the Vainshtein radius the mass of the ghost is of order of the inverse radius, so that the theory cannot be trusted inside this region, not even at the classical level.

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Contents

1. Introduction

In recent years there has been renewed interest in the possibility of giving a mass to the graviton. This idea belongs to a broader class of proposals for modifying gravity at large distances. Besides their theoretical interest, these models could be phenomenologically relevant as possible alternatives to dark matter and dark energy. In this paper we reconsider the issue of the *range of validity* of massive gravity; in particular we concentrate on the stability of classical solutions around massive sources.

The problem we want to address has a long history. Already in the first paper [1] Fierz and Pauli observed that the mass term must be of the form $m_g^2(h^2 - h_{\mu\nu}h^{\mu\nu})$, otherwise a ghost appears in the spectrum, besides the 5 degrees of freedom of the massive spin-2 graviton. A different structure would result in an instability at an energy scale $\sim m_g$. Unfortunately Boulware and Deser showed that this additional degree of freedom propagates when nonlinearities in the action are taken into account [2]. However, from an effective field theory point of view this is not necessarily a problem, until one specifies the scale at which the ghost shows up, i.e. its mass. If this scale is above the UV cutoff Λ of the effective theory this instability can be consistently disregarded.

On the other hand, non-linearities of the classical theory are also the solution to the problem raised by van Dam, Veltman, and Zakharov (vDVZ) [3, 4]: in the linearized theory predictions are not continuous in the limit $m_g \to 0$, because the helicity-0 component of the graviton does not decouple from matter. However, Vainshtein [5] observed that the vDVZ

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discontinuity might not be relevant for macroscopic sources because the linearized approximation around a source of mass M_* breaks down at a distance $R_V = (M_* M_P^{-2} m_g^{-4})^{1/5}$ which diverges for $m_g \to 0$. Classical nonlinearities become important, and the full nonlinear solution could be in perfect agreement with experiments. This is still an open issue in massive gravity, but the Vainshtein effect has been shown to work in a closely related model, the DGP model [6 – 8].

More recently massive gravity has been reconsidered in the effective field theory language, which provides a systematic framework for dealing with quantum effects [9]. For this purpose it is useful to restore the broken diffeomorphism invariance by introducing a set of Goldstone bosons. With this method it is easy to see that the scalar longitudinal component of the graviton becomes strongly coupled at a very low energy scale, much lower than what naively expected by analogy with the spin-1 case. In the Fierz-Pauli theory the strong interaction scale is $\Lambda_5 \sim (m_g^4 M_P)^{1/5}$, which for m_g of order of the present Hubble parameter is $(10^{11}\,\mathrm{km})^{-1}$. By adding to the Fierz-Pauli lagrangian a set of properly tuned interactions of the form $h_{\mu\nu}^n$ the cutoff can be raised up to $\Lambda_3 \sim (m_g^2 M_P)^{1/3} \sim (1000\,\mathrm{km})^{-1}$ [9, 10].

In both cases the theory seems to lose predictivity at very large distances. For instance one can wonder how this strong coupling affects the gravitational potential generated by an astrophysical source. Apparently the potential is uncalculable at distances smaller than 1000 km; but in principle this could not be the case. After all the strong coupling takes place in the Goldstone sector, and inside the Vainshtein radius, if the Vainshtein effect applies, one expects the Goldstone to give a negligible correction to the newtonian potential. Whatever quantum effects take place at the cutoff distance, they could be sufficiently screened from experiments. Nevertheless, without further assumptions, from an effective theory point of view one should include in the lagrangian all the possible operators allowed by the symmetries and weighted by the cutoff. In this case the effective theory loses predictivity at a much larger length scale: these higher dimension operators all become important at a huge distance from the source when evaluated on the classical solution. In the improved theory with cutoff Λ_3 this happens at the corresponding Vainshtein radius $R_V \sim (M_* M_P^{-2} m_g^{-2})^{1/3}$. For the sun R_V is $\sim 10^{16}$ km. This means that we are unable to compute the gravitational potential at distances shorter than R_V without a UV completion. As a consequence there is no range of distances where nonlinear effects can be reliably computed in the effective field theory. If we restrict to the original theory with cutoff Λ_5 the situation is even worse. In this case the infinite tower of higher dimension operators become important at a distance which is parametrically larger than the corresponding Vainshtein radius [9].

The picture looks very similar to the DGP model [11], where the same problems have been pointed out [12]. In the DGP model our world is the 4D boundary of an infinite 5D spacetime. Gravity is described by a standard 5D Einstein-Hilbert action with Planck mass M_5 in the bulk and by an additional 4D Einstein-Hilbert action localized on the boundary, with a much larger Planck mass M_4. The resulting Newton's law is 4-dimensional below the critical length scale $L_{\mathrm{DGP}} = M_4^2/M_5^3$ and 5-dimensional at larger distances. From the 4D viewpoint there is a scalar degree of freedom, the brane bending mode π, whose dynamics is closely related to that of the longitudinal Goldstone boson ϕ of massive gravity. In particular strong interactions show up in the π sector at a tiny energy scale $\Lambda_{\rm DGP} \sim (M_P/L_{\rm DGP}^2)^{1/3} \sim (1000\,{\rm km})^{-1}$ (taking $L_{\rm DGP}$ of order of the present Hubble horizon H_0^{-1}). Also if one includes in the lagrangian all possible operators allowed by the symmetries and suppressed by $\Lambda_{\rm DGP}$, around a heavy source they all become important at the Vainshtein radius $R_V \sim (M_* M_P^{-2} L_{\rm DGP}^2)^{1/3}$ when evaluated on the classical solution.

However all these difficulties depend on assumptions about the UV completion. In the DGP model one can consistently assume a UV completion such that the effective theory is predictive down to distances significantly shorter than $1/\Lambda_{\rm DGP}$. For instance on the surface of the earth the cutoff can be pushed up to $\sim {\rm cm}^{-1}$, not far from the smallest length scale $\sim 100\,\mu{\rm m}$ at which gravity has been experimentally tested. Of course a necessary requirement for this to be possible is the consistency of the classical theory: in particular in DGP no classical instability develops in the π sector for all relevant astrophysical sources and for a large class of cosmological solutions [13]. In this paper we want to study whether the same stability properties hold for massive gravity. This is a basic consistency requirement one has to satisfy before analyzing the theory at the quantum level and looking for a mechanism, analogous to that working in DGP, that can make the theory predictive in a phenomenologically interesting range of scales.

The most convenient way to study the dynamics of the theory is to use the Goldstone formalism, that we review in section 3; for our purposes the main interesting features of the model are encoded in the lagrangian of the longitudinal component ϕ of the Goldstone vector. If we start with a Fierz-Pauli mass term, the dominant interactions for this scalar degree of freedom are cubic self-couplings with 6 derivatives of the form $(\partial^2\phi)^3$. In the presence of a macroscopic source the Goldstone gets a non-trivial configuration $\Phi(x)$; in order to study the stability of such a solution it is necessary to expand the action at quadratic order in the fluctuations around $\Phi(x)$. It is evident that in general, because of the cubic self-coupling, the fluctuations will get a higher-derivative kinetic term. As we discuss in section 2, this signals the presence of a ghost-like instability already at the classical level. In DGP this does not happen: although the π cubic self-coupling has 4 derivatives, its tensorial structure is such that fluctuations around a background get only a 2-derivative kinetic term [12, 13]. Unfortunately, this does not work for the Fierz-Pauli theory. Still, we have a large freedom in choosing the non-linear extension of FP, and one can wonder if it is possible to cancel all higher derivatives terms and end up with a ghost-free theory. Sections 4 and 5 contain the answer: despite the freedom we have, *ghost-like instabilities are unavoidable.*

In the Goldstone language it is also easy to compute the scale at which this instability appears. Even in the most favorable setup in which the cutoff is Λ_3, the ghost enters in the effective field theory at distances from the source parametrically larger than the (already huge) Vainshtein radius R_V. Furthermore, when in approaching the source we reach $r = R_V$ the mass of the ghost has dropped to $1/R_V$! This means that in no way the theory can be extrapolated inside the Vainshtein radius.

The last part of the paper is devoted to discuss how the sickness of the theory is interpreted in the unitary gauge. Clearly we expect the ghost we found to be the troublesome sixth degree of freedom. With the Fierz-Pauli mass term, at quadratic level this degree of freedom does not appear because the trace of the Einstein equations gives a constraint instead of a propagating equation. In section 6 we show that this equation becomes dynamical in the presence of a curved background; we qualitatively estimate in this simple case the mass of this new excitation and the result agrees with the mass we find for the ghost in the Goldstone computation. Then, in section 7, we show in the hamiltonian formalism that there exists no non-linear extension of the Fierz-Pauli theory that can forbid the propagation of the sixth mode in the presence of a slightly curved background. The analysis in the unitary gauge is powerful for counting the number of degrees of freedom, but, unlike the Goldstone analysis, it says nothing about the typical scales of these modes. Also in order to address stability issues one should study the positivity of the hamiltonian. This in general is difficult, and the analysis has been carried out by Boulware and Deser for non-linear extensions of the form $f(h_{\mu\nu}h^{\mu\nu} - h^2)$ [2]. On the contrary the Goldstone analysis concentrates from the very beginning on the strong interacting degree of freedom: all the interesting and troublesome features of the theory are encoded in the dynamics of a single scalar field. This enormously simplifies the analysis.

Recently it has been realized that massive gravity models with Lorentz violating mass terms can be significantly 'healthier' than the traditional Lorentz-invariant theory; it particular they can avoid the vDVZ discontinuity and the strong coupling problem, and they can be free of ghosts [14 − 17]. In this paper we stick to the Lorentz-invariant massive gravity theory.

2. Ghosts from higher derivative kinetic terms

Let us first be very specific about why higher derivative kinetic terms give rise to ghost-like instabilities. Take for instance a massless scalar field ϕ with lagrangian density (note that we are using the $(-,+,+,+)$ signature!)

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 + \frac{a}{2\Lambda^2}(\Box\phi)^2 - V_{\rm int}(\phi)\,, \tag{2.1}$$

where Λ is some energy scale, $a = \pm 1$, and $V_{\rm int}$ is a self-interaction term. We show that, independently of the sign of the second term, the system is plagued by ghosts. To do so we want to reduce to a purely two-derivative kinetic lagrangian, from which we know how to read the stability properties of the system. We therefore introduce an auxiliary scalar field χ and a new lagrangian

$$\mathcal{L}' = -\frac{1}{2}(\partial\phi)^2 - a\,\partial_\mu\chi\partial^\mu\phi - \frac{1}{2}a\,\Lambda^2\,\chi^2 - V_{\rm int}(\phi)\,, \tag{2.2}$$

which reduces exactly to $\mathcal{L}$ once χ is integrated out. $\mathcal{L}'$ is diagonalized by the substitution $\phi = \phi' - a\chi$. We get

$$\mathcal{L}' = -\frac{1}{2}(\partial\phi')^2 + \frac{1}{2}(\partial\chi)^2 - \frac{1}{2}a\,\Lambda^2\,\chi^2 - V_{\rm int}(\phi',\chi)\,, \tag{2.3}$$

which clearly signals the presence of a ghost: χ has a wrong-sign kinetic term. Notice in passing that χ can also be a tachyon, for $a = -1$: in this case χ has exponentially growing modes. But let us neglect this possibility and concentrate on the ghost instability, which is unavoidable. A ghost, unlike a tachyon, is not unstable by itself: its equation of motion is perfectly healthy at the linear level, and does not admit any exponentially growing solution. The problem is that its hamiltonian is negative, so that when couplings to ordinary 'healthy' matter are taken into account (the potential term in our example above) the system is unstable: with zero net energy one can indiscriminately excite both sectors, and this exchange of energy happens spontaneously already at classical level. In a quantum system with ghosts in the physical spectrum this translates into an instability of the vacuum. The decay rate is UV divergent due to an infinite degeneracy of the final state phase space. It is not clear how to cutoff this divergence in a Lorentz invariant way [18].

However the situation is not as bad as it seems: our ghost χ in eq. (2.3) has a (normal or tachyonic) mass Λ, so that it will show up only at energies above Λ, i.e. when the four derivative kinetic term in eq. (2.1) starts dominating over the usual two derivative one. We can consistently use our scalar field theory eq. (2.1) at energies below Λ, and postulate that some new degree of freedom enters at Λ and takes care of the ghost instability. For example, we can add a term $-(\partial\chi)^2$ to eq. (2.2) (for simplicity we stick to the non-tachyonic case $a = +1$ and set $V_{\rm int} = 0$),

$$\mathcal{L}_{\rm UV} = -\frac{1}{2}(\partial\phi)^2 - \partial_\mu\chi\partial^\mu\phi - (\partial\chi)^2 - \frac{1}{2}\,\Lambda^2\,\chi^2\,. \tag{2.4}$$

This drastically changes the high-energy picture, since the resulting lagrangian obtained by demixing now describes two perfectly healthy scalars, one massless and the other with mass Λ. At the same time, at energies below Λ the heavy field χ can be integrated out from $\mathcal{L}_{\rm UV}$, thus giving the starting lagrangian eq. (2.1) up to terms suppressed by additional powers of $(\partial/\Lambda)^2$. This example shows that in principle the ghost instability can be cured by proper new physics at the scale Λ. In other words, eq. (2.1) makes perfect sense as an effective field theory with UV cutoff Λ.

3. The Goldstone action

In this section we briefly re-derive the lagrangian of massive gravity along the lines of [9], i.e. keeping explicit the Goldstone bosons of broken diffeomorphism invariance.

To write down a mass term for gravity, in addition to the full dynamical metric $g_{\mu\nu}$, we have to take a reference fixed metric: for our purposes we will take the Minkowski metric $\eta_{\mu\nu}$. A mass term breaks invariance under general coordinate transformations. However, as it has been shown in ref. [9], one can always restore local coordinate invariance by the Stückelberg trick: in analogy with massive gauge theories one introduces a set of Goldstone fields and requires that they transform non-linearly under a local coordinate transformation. The fundamental object to be used for this purpose is a symmetric tensor $H_{\alpha\beta}$, built in terms of the reference metric, the field describing metric fluctuations $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ and the four Goldstone fields π_μ,

$$H_{\alpha\beta} = h_{\alpha\beta} + \partial_\alpha\pi_\beta + \partial_\beta\pi_\alpha + \partial_\alpha\pi^\gamma\partial_\beta\pi_\gamma\,. \tag{3.1}$$

$H_{\mu\nu}$ transforms as a covariant tensor under local diffeomorphisms $x^\alpha \to x^\alpha + \xi^\alpha$ provided that π_α shifts, $\pi_\alpha \to \pi_\alpha - \xi_\alpha$. As in non-abelian massive gauge theories, since now local coordinate invariance is non linearly realized on the π field, a lagrangian built using H will be valid as an effective theory and its breakdown will appear as the Goldstone sector becoming strongly coupled at some scale Λ. This indeed has been shown in ref. [9]. It is useful to further split the 4 π^α fields into a vector and a scalar as

$$\pi_\mu = A_\mu + \partial_\mu \phi \tag{3.2}$$

together with an additional hidden U(1) gauge invariance for A_μ under which ϕ shifts. Note that since ϕ is a Goldstone boson under this U(1) gauge symmetry and π_μ is a Goldstone boson of broken diffeomorphism invariance ϕ will appear with two derivatives in the lagrangian, as evident from eq. (3.1). A mass term for $h_{\mu\nu}$ can be written down in terms of $H_{\mu\nu}$ as

$$\sqrt{-g} g^{\mu\nu} g^{\alpha\beta} (a H_{\mu\alpha} H_{\nu\beta} + b H_{\mu\nu} H_{\alpha\beta})\,. \tag{3.3}$$

Expanding H using (3.1) one easily realizes that for a generic choice of a, b there is a quadratic term in ϕ containing 4 derivatives. This term signals the presence of a ghost as we have seen in section 2. Only for the Fierz-Pauli choice $a = -b \equiv m_g^2 M_P^2$ this four derivative term exactly cancels. In this case ϕ does not have a kinetic term on its own, but only a kinetic mixing with $h_{\mu\nu}$: $m_g^2 M_P^2 (\partial_\mu \partial_\nu \phi\, h^{\mu\nu} - \Box\phi\, h)$. A conformal rescaling of the metric $h_{\mu\nu} = \hat{h}_{\mu\nu} + m_g^2 \eta_{\mu\nu}\phi$ diagonalizes this mixing and generates a small (i.e. proportional to m_g^2) kinetic term for ϕ besides interactions of the form $\phi(\partial^2\phi)^n$. The smallness of the kinetic term is the origin of the low strong coupling scale as it enhances the ϕ interactions once the fields are canonically normalized.

One can easily see that the most relevant interactions are of the form

$$m^2 M_P^2 (\partial^2 \phi)^3 = \frac{(\partial^2 \phi^c)^3}{M_P m_g^4} \tag{3.4}$$

where ϕ^c is the canonically normalized field. These interactions saturate perturbation theory at the tiny energy scale $E \sim \Lambda_5 \equiv (m_g^4 M_P)^{1/5}$.

One can slightly improve the situation canceling these cubic interactions by adding H^3 terms. Now the most relevant interactions will be of the form $(\partial^2\phi)^4$ and this procedure can be repeated at any order. The dominant interactions will be $(\partial^2\phi)^n$ and once all these are canceled the theory has the cutoff $\Lambda_3 \equiv (m_g^2 M_P)^{1/3}$. In fact after this procedure the most relevant interactions are

$$m_g^2 M_P^2\, (\partial A)^2 (\partial^2\phi)^n \qquad \text{and} \qquad m_g^2 M_P^2\, (\hat{h}_{\mu\nu} + m_g^2 \eta_{\mu\nu}\phi)(\partial^2\phi)^n\,, \tag{3.5}$$

which are weighted by Λ_3 when expressed in terms of canonically normalized fields.

The interaction between matter and gravity is as usual described by the term $\frac{1}{2} h_{\alpha\beta} T^{\alpha\beta}$. The Weyl transformation that demixes $h_{\mu\nu}$ from ϕ thus generates a direct coupling of ϕ to the trace T of the stress-energy tensor. This implies that we will have a non-trivial ϕ background around any astrophysical source. For a classical solution we will have two relevant scales: the first is the Schwarzschild radius R_S, the distance from the source at which linearized gravity breaks down, the second is the Vainshtein radius R_V [5] where nonlinearities for the scalar field ϕ become important. For a source of mass M_* this distance is equal to $1/\Lambda_5(M_*/M_P)^{1/5}$ (which is much larger than R_S). At this scale the term $(\partial^2\phi)^3$ becomes as relevant as the kinetic term whereas all the other nonlinear terms are important only when we reach the Schwarzschild radius so that they can be safely neglected. Therefore the action of ϕ in the presence of sources is given by

$$S = \int d^4x \left\{ 3\phi^c\Box\phi^c + \frac{1}{\Lambda_5^5}\left[(\Box\phi^c)^3 - (\Box\phi^c)(\partial_\mu\partial_\nu\phi^c)^2\right] + \frac{1}{2M_P}\phi^c T \right\}; \tag{3.6}$$

the structure of the trilinear terms can be changed by adding non-linear interactions to the Fierz-Pauli lagrangian.

The situation remains qualitatively unchanged even if the first N $(\partial^2\phi)^n$ interactions are tuned to zero. The first non-vanishing term, $(\partial^2\phi)^{N+1}$, will set the Vainshtein radius, while higher order terms will become relevant again at the Schwarzschild radius.

4. Massive gravity in the presence of a source

Let us consider the lagrangian eq. (3.6) in the presence of a macroscopic source, like the Sun. This induces a classical background $\Phi(x)$, solution of the ϕ equation of motion. To study the stability of this solution we expand the lagrangian at the quadratic order in the fluctuation $\varphi \equiv \phi - \Phi$. The result is schematically of the form

$$\mathcal{L}_\varphi = -(\partial\varphi^c)^2 + \frac{(\partial^2\Phi^c)}{{\Lambda_5}^5}(\partial^2\varphi^c)^2\,, \tag{4.1}$$

i.e. the background gives a four-derivative contribution to the φ kinetic term. As discussed in section 2, this results in the appearance of a ghost with an x-dependent mass

$$m^2_{\rm ghost}(x) \sim \frac{{\Lambda_5}^5}{\partial^2\Phi^c(x)}\,. \tag{4.2}$$

Remember that we are dealing with an effective theory with a tiny UV cutoff Λ_5, therefore we should not worry until the mass of the ghost drops below Λ_5. In approaching the source from far away, this happens at a distance $R_{\rm ghost}$ from the source such that $\partial^2\Phi^c \sim {\Lambda_5}^3$. Unfortunately this is a huge distance, parametrically larger than the (already huge) Vainshtein radius R_V. In fact for a source of mass M_* at distances $r \gg R_V$ the background field goes as $\Phi^c(r) \sim (M_*/M_P)\cdot 1/r$, so that

$$R_{\rm ghost} \sim \frac{1}{\Lambda_5}\left(\frac{M_*}{M_P}\right)^{1/3} \gg R_V^{(5)} \sim \frac{1}{\Lambda_5}\left(\frac{M_*}{M_P}\right)^{1/5}. \tag{4.3}$$

Therefore the ghost is going to show up in an extremely weak background field, when the latter is still in its linear regime.

Inside $R_{\rm ghost}$, in the spirit of section 2, one is forced to postulate that additional physics lighter than the local ghost mass cures the instability, that is the cutoff must be lowered from Λ_5 to $m_{\rm ghost}(x)$. A byproduct of this in general would be that interactions strengthen, being weighted by the new cutoff scale rather than by Λ_5. But let us optimistically assume that, instead, the only effect of this new physics is to cure the ghost instability. However, when the local ghost mass is of order of the inverse distance from the source there is no way of proceeding further without specifying the UV completion of the theory, since the background itself has a typical length scale of order of the UV cutoff. One can easily check that this happens at the Vainshtein radius R_V. *There is no sense in which one can trust the classical solution below* R_V. Since one can hope to recover General Relativity only in the region inside R_V, where non-linear effects can hide the scalar (Vainshtein effect), this also means that *General Relativity is nowhere a good approximation.*

Notice that in the DGP model the dominant interaction of the Goldstone has the form $\Box\pi(\partial\pi)^2$, which suggests the same problem we are facing, as there are two derivatives acting on one of the π's. Nevertheless, in the equation of motion terms with more than two derivatives acting on a single field cancel out and one is left with a (non-linear) second order differential equation [13].[1] The same cannot happen in our case since there are too many derivatives: the contribution of the trilinear term to the equation of motion is a sum of terms with 2 ϕ's and 6 derivatives. In any term there is at least one ϕ carrying more than two derivatives.

One is thus led to consider the possibility of eliminating the unwanted trilinear interaction of the Goldstone by adding appropriate cubic terms in $H_{\mu\nu}$ to the Fierz-Pauli lagrangian eq. (3.3). The three independent contractions are H^3, $H(H_{\mu\nu})^2$, and $(H_{\mu\nu})^3$, where the last stands for the cyclic contraction of the indices. These contain interaction terms for the Goldstone of the form $(\partial^2\phi)^3$ which, for the proper choice of coefficients, cancel the trilinear interaction of eq. (3.6). However, in this way one introduces further quartic interactions $(\partial^2\phi)^4$ on top of those already present in the Fierz-Pauli mass term, because of the non-linear relation between $H_{\mu\nu}$ and ϕ of eq. (3.1). These are problematic for exactly the same reason as before, and the same problem shows up at any order: an interaction term of the form $(\partial^2\phi)^n$ evaluated around a background gives a contribution to the equation of motion for the fluctuations with too many derivatives. This signals the presence of a ghost instability.[2] Again one can check that the cutoff (i.e. the ghost

[1] Equivalently, working at the level of the lagrangian, one can expand the interaction term $\Box\pi(\partial\pi)^2$ to second order in the fluctuation φ around a background π_b. The worrisome term is $\Box\varphi\partial_\mu\varphi\,\partial^\mu\pi_b$, since it has 3 derivatives acting on the φ's. But by integration by parts one can shift one derivative from the fluctuations to the background, thus obtaining an ordinary 2-derivative kinetic term for the fluctuations (whose positivity must however be checked) [12, 13].

[2] The reader could wonder if there exists a choice of coefficients such that terms with 4 derivatives on a single field cancel in the equation of motion. In this case one would be left only with 3-derivative terms and our conclusions should be modified. But this is not the case: setting to zero all 4 derivative terms leads also to the cancellation of those with 3 derivatives. To see this, consider the most general interaction lagrangian of n-th order, $\mathcal{L}^{(n)} = \Gamma^{\alpha_1\beta_1\cdots\alpha_n\beta_n}\,\partial_{\alpha_1}\partial_{\beta_1}\phi\cdots\partial_{\alpha_n}\partial_{\beta_n}\phi$, where Γ is a tensor constructed with the metric $\eta_{\mu\nu}$. Given the structure of contractions, without loss of generality we can choose $\Gamma^{\alpha_1\beta_1\cdots\alpha_n\beta_n}$ to be symmetric under $\alpha_i \leftrightarrow \beta_i$ and $(\alpha_i,\beta_i)\leftrightarrow(\alpha_j,\beta_j)$. Then, for symmetry reasons, the contribution of $\mathcal{L}^{(n)}$ to the ϕ equation of motion is

$$\Gamma^{\alpha_1\beta_1\cdots\alpha_n\beta_n}\left[A_n\,(\partial_{\alpha_1}\partial_{\beta_1}\partial_{\alpha_2}\partial_{\beta_2}\phi)(\partial_{\alpha_3}\partial_{\beta_3}\phi)+B_n\,(\partial_{\beta_1}\partial_{\alpha_2}\partial_{\beta_2}\phi)(\partial_{\alpha_1}\partial_{\alpha_3}\partial_{\beta_3}\phi)\right]\partial_{\alpha_4}\partial_{\beta_4}\phi\cdots\partial_{\alpha_n}\partial_{\beta_n}\phi\,, \quad (4.4)$$

where A_n, B_n are combinatoric factors. The four-derivative term (the first in brackets) identically vanishes mass) becomes of order of the inverse radius at the new Vainshtein scale (always defined as the distance from the source at which non-linearities become relevant). Hence one never recovers General Relativity.

The only possibility is therefore to concentrate on theories in which all the interactions of the form $(\partial^2\phi)^n$ are set to zero by properly choosing infinitely many coefficients. We can look at this procedure as an extension at non-linear order of the Fierz-Pauli choice for the mass term which, as discussed in section 3, leads to the cancellation of the $(\partial^2\phi)^2$ terms. In the next section we prove that also in this case we cannot avoid higher derivative kinetic terms for the fluctuations of the field ϕ around a non trivial background: *ghost-like instabilities are unavoidable.*

5. Ghost instabilities are unavoidable

The cancellation of the $(\partial^2\phi)^n$ interactions has also been considered as a way to raise the strong interaction scale to $\Lambda_3 = (m_g^2 M_P)^{1/3} \gg \Lambda_5$ [9, 10]. In fact, after the cancellation, the leading interactions are of the form

$$m_g^2 M_P^2\,(\partial A)^2(\partial^2\phi)^n \qquad \text{and} \qquad m_g^2 M_P^2\,(\hat{h}_{\mu\nu} + m_g^2\eta_{\mu\nu}\phi)(\partial^2\phi)^n\,; \tag{5.1}$$

when the fields are canonically normalized all these terms are suppressed by the scale Λ_3, while additional interactions are weighted by higher scales. Correspondingly the Vainshtein radius now shrinks to $R_V^{(3)} = 1/\Lambda_3(M_*/M_P)^{1/3} \ll R_V^{(5)}$ as the original leading interactions have been canceled. At this radius all non-linear terms of the form $(\hat{h}_{\mu\nu} + m_g^2\eta_{\mu\nu}\phi)(\partial^2\phi)^n$ become relevant; on the other hand there are no terms linear in A in the lagrangian, so that A is sourced neither by matter nor by the other fields. Interactions involving this field are therefore irrelevant for our purposes, and can be consistently neglected.

The theory we are describing is not unique: there are different possible choices of coefficients that cancel all the interactions $(\partial^2\phi)^n$. We can easily see why. Let us start with the canonical Fierz-Pauli mass term[3] $\mathcal{L}_2 = \sqrt{-g}\,\left([H^2] - [H]^2\right)$; since $H_{\mu\nu} = h_{\mu\nu} + 2\,\partial_\mu\partial_\nu\phi + \partial_\mu\partial_\alpha\phi\,\partial_\nu\partial^\alpha\phi$ (setting $A_\mu = 0$), $\mathcal{L}_2$ contains $(\partial^2\phi)^3$ interactions. We can cancel them adding an appropriate combination of terms cubic in H, $\mathcal{L}_3 = \sqrt{-g}\,\left(\frac{1}{2}[H][H^2] - \frac{1}{2}[H^3]\right)$. But at this point we can still add, with an arbitrary overall coefficient α_3, the expression

$$\mathcal{L}_3^{\mathrm{TD}} = \sqrt{-g}\,\left(3[H][H^2] - [H]^3 - 2[H^3]\right), \tag{5.2}$$

because it gives $(\partial^2\phi)^3$ terms in the combination

$$(\Box\phi)^3 - 3\,\Box\phi\,(\partial_\mu\partial_\nu\phi)^2 + 2\,(\partial_\mu\partial_\nu\phi)^3\,, \tag{5.3}$$

only if the totally symmetric part of $\Gamma^{\alpha_1\beta_1\cdots\alpha_n\beta_n}$ in the first four indices does, $\Gamma^{(\alpha_1\beta_1\alpha_2\beta_2)\cdots\alpha_n\beta_n} = 0$. In this case, given the symmetries of Γ, it is straightforward to check that $\Gamma^{\alpha_1(\beta_1\alpha_2\beta_2)\cdots\alpha_n\beta_n} = \Gamma^{(\alpha_1\beta_1\alpha_2\beta_2)\cdots\alpha_n\beta_n} = 0$. This eliminates the 3 derivative term (the second in brackets) as well, i.e. eliminates the contribution of $\mathcal{L}^{(n)}$ to the ϕ equation of motion altogether.

[3]We use the notation $[H] = g^{\mu\nu}H_{\mu\nu}$, $[H^2] = g^{\mu\nu}g^{\alpha\beta}H_{\mu\alpha}H_{\nu\beta}$, and its straightforward generalization to higher orders.

which is a total derivative (hence the superscript 'TD') and thus does not contribute to the equation of motion. Now $\mathcal{L}_2 + \mathcal{L}_3 + \alpha_3 \mathcal{L}_3^{\rm TD}$ contains $(\partial^2\phi)^4$ interactions and we can repeat the procedure. Fourth order terms are canceled by

$$\mathcal{L}_4 = \sqrt{-g}\,\frac{1}{16}\Big[(5+24\alpha_3)[H^4]-(1+12\alpha_3)[H^2]^2-(4+24\alpha_3)[H][H^3]+12\alpha_3[H^2][H]^2\Big]. \quad (5.4)$$

Again we have the possibility to introduce a second arbitrary coefficient, α_4, in front of the 'total derivative' term

$$\mathcal{L}_4^{\rm TD} = \sqrt{-g}\left([H]^4 - 6[H^2][H]^2 + 8[H^3][H] + 3[H^2]^2 - 6[H^4]\right), \quad (5.5)$$

and we can go on at higher orders until all self-couplings are removed. Notice that these total derivative terms $\mathcal{L}_n^{\rm TD}$ are the higher order analogue of the Fierz-Pauli mass term: they are combinations of $h_{\mu\nu}$ interactions that reduce to a total derivative when expressed in terms of $\partial_\mu\partial_\nu\phi$ and therefore do not contribute to high-derivative terms in ϕ. One can check that there is one of such combinations per order, but for our purposes we will need them only up to fourth order.[4]

We now want to see if it is possible to get a kinetic structure without ghosts for the fluctuations around a background. Note that this theory has potentially dangerous terms of the form $h_{\mu\nu}(\partial^2\phi)^n$. We could hope that the large freedom we have in the choice of higher-order terms, parameterized by the coefficients $\alpha_3, \alpha_4, \ldots$, helps us to obtain a ghost-free theory. Unfortunately, this is not the case and the main reason is that the number of possible contractions of $H_{\mu\nu}$ grows very fast and soon we cannot cancel all the dangerous kinetic terms. Let us see how this works explicitly. We call $h^b_{\mu\nu}$ and ϕ^b the background fields,[5] and we study the quadratic lagrangian for the scalar fluctuation φ. Actually, instead of working at the level of the lagrangian, the most direct approach is to look at the equations of motion linear in the fluctuations, because in this case there is no ambiguity coming from integration by parts. We have to check whether all the terms with more than 2 derivatives on φ can cancel in the equations of motion.

We start with the lagrangian terms *cubic* in the fields: they can come only from $\mathcal{L}_2 + \mathcal{L}_3 + \alpha_3\mathcal{L}_3^{\rm TD}$. Only terms schematically of the form $h(\partial^2\phi)^2$ can give a ghost; terms with a higher number of $h_{\mu\nu}$ have no more than 2 derivatives. Using the explicit expression of the lagrangian, it is immediate to verify that *i)* the terms in the e.o.m. with more than 2 derivatives on φ cancel already with $\alpha_3 = 0$, and *ii)* they cancel also when originating

[4]It is easy to check that at n-th order a total derivative term is given by

$$\sum_\pi (-1)^\pi \eta^{\alpha_1\pi(\beta_1)}\cdots\eta^{\alpha_n\pi(\beta_n)}\,\partial_{\alpha_1}\partial_{\beta_1}\phi\cdots\partial_{\alpha_n}\partial_{\beta_n}\phi\,, \quad (5.6)$$

where the sum runs over all permutations π of the β indices, and $(-1)^\pi$ is the parity of the permutation. To prove that this combination is the only total derivative term at a given order assume that there are two of them. One then could construct a total derivative term that does not contain, say, $(\Box\phi)^n$. Imposing that the contribution to the field equations is zero, it is straightforward to show that also all the other terms vanish.

[5]Remember that the field $h_{\mu\nu}$ is the graviton *before* the Weyl rescaling that demixes it from ϕ: $h_{\mu\nu} = \hat{h}_{\mu\nu} + m_g^2\eta_{\mu\nu}\phi$.

from $\mathcal{L}_3^{\rm TD}$ alone. We conclude that at the cubic level in $H_{\mu\nu}$ there are no higher derivative kinetic terms for φ and the parameter α_3 can still be varied arbitrarily.

What happens with the *quartic* lagrangian? Let us consider the interactions $h_{\mu\nu}(\partial^2\phi)^3$; now we must include also $\mathcal{L}_4$ (eq. (5.4)), and for simplicity we start with $\alpha_4 = 0$. Again it is straightforward to write down the equations of motion linear in φ. The terms with more than two derivatives on the fluctuation can be divided into two classes: those containing $(h^b)^\mu{}_\mu$ and those in which $(h^b)_{\mu\nu}$ is contracted with derivatives of ϕ. Either class must cancel independently of the other, since, given the different tensor structure, there is no possibility of cross-cancellation between the two. In the e.o.m. the terms belonging to the first class automatically cancel. They come from a piece of the lagrangian that is precisely $4\alpha_3\,[h]$ times the 'total derivative' combination eq. (5.3). The remaining dangerous kinetic terms, which belong to the second class, come from the lagrangian[6]

$$(8+72\alpha_3)\left([h\,\phi^3]-[h\,\phi^2][\phi]\right)-36\alpha_3\left([h\,\phi][\phi^2]-[h\,\phi][\phi]^2\right). \tag{5.7}$$

Can we add now $\alpha_4\mathcal{L}_4^{\rm TD}$ (eq. (5.5))? Yes, since this does not reintroduce terms of the first class (i.e., containing $(h^b)^\mu{}_\mu$) in the equations of motion. Then, can we choose α_4 to get rid of the contributions coming from eq. (5.7)? Unfortunately the answer is negative. In fact, expanding $\alpha_4\mathcal{L}_4^{\rm TD}$,

$$\alpha_4\mathcal{L}_4^{\rm TD}\supset -192\alpha_4\left([h\,\phi^3]-[h\,\phi^2][\phi]\right)+96\alpha_4\left([h\,\phi][\phi^2]-[h\,\phi][\phi]^2\right), \tag{5.8}$$

one immediately sees that either the first or the second pair of terms in eq. (5.7) (but not both) can be canceled by properly choosing α_3 and α_4. The other pair gives a non-zero, four derivative contribution to the equation of motion for the fluctuation φ. The number of possible tensor structures is bigger than the freedom we have in the lagrangian. This completes the proof that *massive gravity around a generic background cannot have a purely two-derivative kinetic term for* φ.

In the Λ_3 theory the local mass of the ghost goes as

$$m^2_{\rm ghost}(x)\sim\frac{{\Lambda_3}^6}{\Phi^c(x)\,\partial^2\Phi^c(x)}\,, \tag{5.9}$$

so that the ghost enters in the effective theory at a distance from the source much bigger than the Vainshtein radius,

$$R_{\rm ghost}\sim\frac{1}{\Lambda_3}\left(\frac{M_*}{M_P}\right)^{1/2}\gg R_V^{(3)}\sim\frac{1}{\Lambda_3}\left(\frac{M_*}{M_P}\right)^{1/3}. \tag{5.10}$$

These results should be compared with their analogues in the Λ_5 theory, eqs. (4.2) and (4.3). Again at the Vainshtein scale the mass of the ghost is of order of the inverse radius and we cannot proceed further towards the source.

There is a final subtlety that needs to be addressed. Is the presence of a 4-derivative kinetic term enough to claim that there is a ghost? After all, the argument of section 2

[6] Inside brackets with ϕ we indicate the matrix $\partial_\mu\partial_\nu\phi$. For example the term $[h\,\phi][\phi]^2$ should be read as $\eta^{\mu\alpha}\eta^{\nu\beta}\,h_{\mu\nu}\,\partial_\alpha\partial_\beta\phi\,(\Box\phi)^2$.

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strictly applies only to a Lorentz-invariant background. It is clear that if the derivatives acting on the fluctuation φ are contracted with a background tensor field different from $\eta_{\mu\nu}$ the situation can be very different. For instance, if the quadratic lagrangian for φ involves 4 space derivatives but only 2 time derivatives there is no room for an independent propagating extra scalar (χ, in the language of section 2), and thus there is no ghost, provided that the $\dot{\varphi}^2$ term has the healthy sign. Indeed, in most astrophysical situations macroscopic sources have non-relativistic velocities $v \ll 1$; the background ϕ^b field they generate is therefore essentially constant in time, its time derivatives being suppressed by positive powers of v with respect to its spatial gradients. This, in a term like $\partial^\mu\partial^\nu\phi^b\,\partial_\mu\partial_\rho\varphi\partial^\rho\partial_\nu\varphi$ for instance, can suppress the magnitude of terms with 4 time derivatives. Although this is a parametric increase of the ghost mass, for typical astrophysical velocities $v \sim 10^{-4} - 10^{-3}$ the regime of validity of the theory is not significantly widened: inside the Vainshtein radius the theory breaks down at $r \sim R_V^{(5)} v^{4/5} \sim (10^{-3} - 10^{-2}) R_V^{(5)}$, where the ghost mass becomes of order of the inverse radius.

6. Unitary gauge description

In the previous section we argued in the Goldstone language that Fierz-Pauli massive gravity (together with all its infinitely many higher order extensions) is unavoidably plagued by ghosts around a tinily curved background. We now want to see how the sickness of the theory is interpreted in the unitary gauge. In order to take into account the presence of a (slightly) curved background we need to consider the field equations at second order in $h_{\mu\nu}$,

$$G_{\mu\nu} + m_g^2[a\,h_{\mu\nu} + b\,h\eta_{\mu\nu} + \mathcal{O}(h_{\mu\nu}^2)] = 0\,, \tag{6.1}$$

where the quadratic terms come both from the mass term (that we take generic for the moment) and from additional higher order interactions. As it is well known, the invariance of the Einstein-Hilbert action under diffeomorphisms $g_{\mu\nu} \to g_{\mu\nu} + \nabla_\mu\epsilon_\nu + \nabla_\nu\epsilon_\mu$,

$$\delta S_{\rm EH} = 2\int d^4x\,\frac{\delta S_{\rm EH}}{\delta g_{\mu\nu}}\nabla_\mu\epsilon_\nu = -2\int d^4x\sqrt{-g}\,\epsilon_\nu\,\nabla_\mu\left(\frac{1}{\sqrt{-g}}\frac{\delta S_{\rm EH}}{\delta g_{\mu\nu}}\right) = 0\,, \tag{6.2}$$

implies the contracted Bianchi identities $\nabla^\mu G_{\mu\nu} = 0$. As a consequence, from eq. (6.1) we get the four constraints

$$\nabla^\mu[a\,h_{\mu\nu} + b\,h\eta_{\mu\nu} + \mathcal{O}(h_{\mu\nu}^2)] = 0\,, \tag{6.3}$$

which reduce to six the number of propagating components of $h_{\mu\nu}$. The presence of these constraints is ensured by the gauge-invariance of the 'kinetic' action. This is in complete analogy to what happens in the theory of a massive vector particle, where the gauge invariance of the kinetic term implies the constraint $\partial_\mu A^\mu = 0$, thus reducing to three the number of propagating degrees of freedom.

If we now restrict to a linear analysis, for the particular Fierz-Pauli choice of the mass term ($b = -a$) we have an additional constraint equation. In fact the linearized Einstein tensor satisfies

$$G^{\ell\,\mu}{}_{\mu} = \partial^\mu\partial^\nu(h_{\mu\nu} - h\eta_{\mu\nu})\,, \tag{6.4}$$

so that eq. (6.3) forces ${G^\mu}_\mu$ to vanish at linear order, and thus the trace of the Einstein equations eq. (6.1) becomes a constraint for the trace mode,

$$h = 0\,. \tag{6.5}$$

In the end one is left with five propagating degrees of freedom, the correct number for a massive spin-2 particle. However, it is clear that this last constraint is fundamentally different from the previous four, since, unlike them, is not ensured by any symmetry, but it is based on a precise tuning in the structure of the mass term and on the identity eq. (6.4), valid at linear order. It is then natural to expect that it does not survive in a curved background. The ghost we found in the Goldstone language is nothing but this hidden sixth mode that, although constrained in flat space, starts propagating around a non-zero background. Let us check that the energy scale at which the additional mode appears is indeed the same in the two descriptions.

At quadratic order ${G^\mu}_\mu$ will not vanish anymore on the equations of motion, instead it will be of the form ${G^\mu}_\mu \sim \partial^2 h^2_{\mu\nu}$. Therefore at quadratic order the constraint eq. (6.5) becomes a dynamical equation,

$$\mathcal{O}(\partial^2 h^2_{\mu\nu}) + m_g^2[h + \mathcal{O}({h_{\mu\nu}}^2)] = 0\,. \tag{6.6}$$

If we now write $h_{\mu\nu}$ as the sum of the background field and fluctuations around it we see that the equation above describes the propagation of a mode with mass

$$m^2_{6^{\rm th}\text{-mode}}(x) \sim \frac{m_g^2}{\mathcal{H}_{\mu\nu}(x)}\,, \tag{6.7}$$

where $\mathcal{H}_{\mu\nu}$ is the background metric. Notice that in the limit of zero background the mass goes to infinity and the mode decouples, as expected from the linear analysis.[7]

In order to compare this with our results in the Goldstone language we must relate the background $\mathcal{H}_{\mu\nu}(x)$ in the unitary gauge to the Goldstone background $\Phi^c(x)$. This is easily done by getting rid of the Goldstone, i.e. by performing a gauge transformation with parameter $\epsilon_\mu = \frac{1}{m_g^2 M_{\rm Pl}}\partial_\mu \Phi^c$. Considering for definiteness the case of a spherical source, the background in unitary gauge is therefore the sum of the usual Schwarzschild solution of GR $\mathcal{H}^{\rm S}_{\mu\nu}(r)$ (corrected by the kinetic mixing with Φ, hence the vDVZ discontinuity) and of the pure-gauge longitudinal contribution $\frac{1}{m_g^2 M_{\rm Pl}}\partial_\mu\partial_\nu\Phi^c(r)$. Both $\mathcal{H}^{\rm S}_{\mu\nu}$ and $\frac{1}{M_{\rm Pl}}\Phi^c$ scale as R_S/r outside the Vainshtein radius, so that the pure-gauge contribution coming from the Goldstone is the dominant one in unitary gauge,

$$\mathcal{H}_{\mu\nu}(r) \sim \frac{1}{m_g^2 M_{\rm Pl}}\partial_\mu\partial_\nu\Phi^c(r) \sim \frac{1}{(m_g r)^2}\frac{R_S}{r} \gg \mathcal{H}^{\rm S}_{\mu\nu}(r)\,. \tag{6.8}$$

[7] A similar result has been obtained in ref. [19], where it has been interpreted as an instability of arbitrarily short time-scale of Minkowski space. Note however that from our effective theory point of view when the instability time-scale gets shorter than the cutoff the presence of the instability entirely depends on the UV completion, as stressed in section 2.

This means that the mass of the 'sixth mode' eq. (6.7) is dominated, as expected, by the Φ background,

$$m^2_{6^{\text{th}}\text{-mode}}(x) \sim \frac{m_g^4 M_{\text{Pl}}}{\partial_\mu \partial_\nu \Phi^c} \sim \frac{{\Lambda_5}^5}{\partial^2 \Phi^c}\,, \tag{6.9}$$

which is exactly the mass of the ghost eq. (4.2) we found in the Goldstone language!

7. The sixth mode in the ADM formalism

As we did in the Goldstone language we now show directly in unitary gauge that it is not possible to forbid the propagation of the sixth mode by adding properly tuned higher order terms to the action. We do this in the hamiltonian formalism, where the counting of degrees of freedom is explicit. Let us introduce the ADM variables $\{N,\, N_j,\, \hat{g}_{ij}\}$ [20],

$$N \equiv \frac{1}{\sqrt{-g^{00}}}\,, \qquad N_j \equiv g_{0j}\,, \tag{7.1}$$

and $\hat{g}_{ij}$ is the 3D metric induced on spatial hypersurfaces of constant t. It is well known that in GR N and N_j are not dynamical fields: the Einstein-Hilbert lagrangian does not contain their time derivatives, so that their conjugate momenta vanish identically. Moreover in the hamiltonian they appear linearly, as Lagrange multipliers. Therefore their equations of motion are really constraint equations for the other degrees of freedom $\hat{g}_{ij}$ and their conjugate momenta π^{ij}, rather than equations for N and N_j. These are the so-called momentum and hamiltonian constraints. The hamiltonian system is thus reduced to two independent (q,p) pairs, which describe the two graviton modes.

If we now perturb GR by adding a mass term for $h_{\mu\nu}$, or generic interactions involving only $h_{\mu\nu}$ and not its derivatives, the lagrangian still does not contain time derivative of N and N_j. However in general N and N_j now do not appear linearly in the hamiltonian. In this case their equations of motion are now determining their value rather than constraining other degrees of freedom. This raises to six the number of d.o.f. of massive gravity. The Fierz-Pauli tuning of the mass term precisely corresponds to setting to zero the N^2 term in the action, so that (at quadratic order) the variation with respect to N still gives a constraint equation, eliminating the unwanted 'sixth mode'. We want to see if similar tunings can work at all orders.

Expressed in terms of the ADM variables the metric fluctuation $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ takes the form

$$h_{\mu\nu} = \begin{pmatrix} 1 - N^2 + \hat{g}^{kl}\, N_k N_l & N_i \\ N_j & h_{ij} \end{pmatrix}, \tag{7.2}$$

where $h_{ij} \equiv \hat{g}_{ij} - \delta_{ij}$, and $\hat{g}^{ij}$ is the inverse of $\hat{g}_{ij}$. We are going to work perturbatively in $\delta N \equiv N - 1$, N_j and h_{ij}, so from now on spatial indices are contracted with the euclidean 3D metric δ_{ij}. Notice that, given the non-linear relation between $h_{\mu\nu}$ and the ADM variables, a generic n-th order expression in $h_{\mu\nu}$ also contributes to orders higher than n when expressed in ADM variables.

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Quadratic terms. At quadratic order in $h_{\mu\nu}$ the most general lagrangian is[8]

$$\mathcal{L}_2 = a_2[h^2] + b_2[h]^2\,. \tag{7.3}$$

By plugging eq. (7.2) in this expression we find the term proportional to δN^2,

$$\mathcal{L}_2 \supset 4(a_2 + b_2)\,\delta N^2\,, \tag{7.4}$$

hence the Fierz-Pauli tuning $b_2 = -a_2$ to set it to zero. The coefficient a_2 fixes the mass of the graviton, and for our purposes we can take it to be 1. At quadratic level (in ADM variables) the problem is solved, but $\mathcal{L}_2$ also contributes to third and fourth order terms; in particular it contains a term $-2h_{ii}\,\delta N^2$, which upon variation with respect to δN gives rise to an equation for δN itself rather than to a constraint. We are therefore forced to introduce cubic terms in $h_{\mu\nu}$.

Cubic terms. The cubic lagrangian is

$$\mathcal{L}_3 = a_3[h^3] + b_3[h][h^2] + c_3[h]^3\,. \tag{7.5}$$

Cubic terms in ADM variables come both from this and from $\mathcal{L}_2$. In particular, those involving more than one δN are

$$\mathcal{L}_2 + \mathcal{L}_3 \supset (12c_3 + 4b_3 - 2)h_{ii}\,\delta N^2 + 8(a_3 + b_3 + c_3)\,\delta N^3\,. \tag{7.6}$$

We can set both of them to zero by choosing

$$a_3 = 2c_3 - \frac{1}{2}\,, \qquad b_3 = \frac{1}{2} - 3c_3\,. \tag{7.7}$$

This agrees with what we found in the Goldstone language, and again the coefficient c_3 is still undetermined. Now we are forced to introduce quartic terms in $h_{\mu\nu}$ to cancel undesired quartic terms containing δN^n coming both from $\mathcal{L}_2$ and $\mathcal{L}_3$.

Quartic terms. The quartic lagrangian is

$$\mathcal{L}_4 = a_4[h^4] + b_4[h][h^3] + c_4[h^2]^2 + d_4[h]^2[h^2] + e_4[h]^4\,. \tag{7.8}$$

Again, we are only interested in terms involving powers of δN larger than 1. For symmetry reasons these must be of the form

$$\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 \supset \left[A\,h_{ii}^2 + B\,h_{ij}h_{ij} + C\,N_jN_j\right]\delta N^2 + D\,h_{ii}\,\delta N^3 + E\,\delta N^4\,. \tag{7.9}$$

After a straightforward but lengthy computation we find the relationship between the coefficients $(A,\ldots,E)$ and $(c_3, a_4, \ldots, e_4)$,

$$\begin{pmatrix} A \\ B \\ C \\ D \\ E \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 & 0 & 4 & 24 \\ -3 & 0 & 0 & 8 & 4 & 0 \\ 0 & -16 & -12 & -16 & -8 & 0 \\ 0 & 0 & 8 & 0 & 16 & 32 \\ 0 & 16 & 16 & 16 & 16 & 16 \end{pmatrix} \cdot \begin{pmatrix} c_3 \\ a_4 \\ b_4 \\ c_4 \\ d_4 \\ e_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \\ 2 \\ 0 \end{pmatrix}. \tag{7.10}$$

[8] In this section all contractions are done with the flat metric $\eta_{\mu\nu}$ and there is no $\sqrt{-g}$ in the action. Different conventions are equivalent in unitary gauge: they just reshuffle the coefficients in the expansion in $h_{\mu\nu}$.

We would like to set the vector $(A, \ldots, E)$ to zero. Since we have 6 free coefficients $(c_3, a_4, \ldots, e_4)$ to choose and only 5 conditions to satisfy, one naively expects this to be possible and one of the free coefficients to remain undetermined. On the contrary, it is impossible. This is because the matrix above has rank 4 and the space spanned by it does not contain the inhomogeneous term. There is no way of choosing $(c_3, a_4, \ldots, e_4)$ to make the unwanted expression eq. (7.9) vanish.

In summary, we tried to tune all interactions $h^n_{\mu\nu}$ in order to keep the hamiltonian linear in N (or equivalently in δN), this to ensure the presence of a constraint equation that eliminates the troublesome sixth degree of freedom. We found that when fourth order terms are taken into account this tuning is impossible. This agrees with our result in the Goldstone language, namely that it is impossible to tune fourth order interactions to avoid the propagation of a ghost.

Notice however that the ADM analysis we sketched in this section is useful to explicitly count the number of degrees of freedom but, unlike the Goldstone approach, gives us no clue on the typical mass of these modes. From the effective field theory point of view this additional information is crucial: a massive mode with a mass above the cutoff can be consistently discarded, even if it is a ghost or a tachyon. Also we have not checked in this language that the sixth mode is a ghost. To do so one should compute the hamiltonian and see that it is not positive definite. This approach is rather cumbersome and it has been carried out in ref. [2] only for a limited set of non-linear terms, namely functions of the Fierz-Pauli mass term: $f(h^2_{\mu\nu} - h^2)$.

8. Concluding remarks

It this paper we have shown that in massive gravity the classical solutions around a source are plagued by ghost instabilities. This holds for any choice of the non-linear terms one can add to the Fierz-Pauli action. It is known that massive gravity is an effective field theory whose UV cut-off can be pushed at most up to $\Lambda_3 = (m_g^2 M_P)^{1/3}$. In this optimal case the ghost instability enters in the effective field theory at a distance from the source $R_{\rm ghost} \sim 1/\Lambda_3 \cdot (M_*/M_P)^{1/2}$. This distance is huge, much bigger than the Vainshtein radius $R_V \sim 1/\Lambda_3 \cdot (M_*/M_P)^{1/3}$. For instance taking $m_g \sim H_0$, for an astrophysical source like the Sun $R_{\rm ghost} \sim 10^{22}$ km, of order of the cosmological horizon! One could optimistically postulate that new physics enters at energies of order of the (local) ghost mass and cures the instability; even under this hypothesis, when the mass of the ghost becomes of order of the inverse distance from the source there is no way of proceeding further without specifying the UV completion of the theory. This happens at the Vainshtein radius (of order 10^{16} km for the Sun); as the vDVZ discontinuity can be cured only inside the Vainshtein radius, General Relativity is never a good approximation.

One could argue that the very low cut-off of massive gravity is already bad enough to disregard the theory. For instance, if one assumes that the theory has a generic series of higher dimension operators suppressed by the cut-off, these will become important when evaluated on a classical solution at huge length scales, parametrically bigger than the inverse cut-off [9]. This implies that it is impossible to calculate the classical solution around a source without specifying the UV completion. However this problem depends on the high energy completion of the theory, and its precise formulation is subtler than what one can argue at first sight. For example in the DGP model, which in this respect is very similar to massive gravity, one can make consistent assumptions on the higher dimension terms which make predictions independent of the UV completion [13]. Of course a prerequisite for this to work is that the classical theory is free from pathologies and instabilities. As we have shown this is not the case in massive gravity: ghost instabilities are unavoidable. The theory is inconsistent already at the classical level, before taking into account quantum effects.

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